Name: Solutions

Instructor: ____________________

Math 10560, Final Exam:
May 7, 2014

- The Honor Code is in effect for this examination, including keeping your answer sheet under cover.
- No **calculators** should be used during the exam.
- **Turn off and put away all cell-phones** and similar electronic devices.
- **Head phones** are not allowed.
- **Put away all notes and formula sheets** where they cannot be viewed.
- The exam lasts for two hours.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 15 pages of the test.
- Hand in the entire exam.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(a)	(b)	(c)	(d)	(e)	15.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)	16.	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)	17.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)	18.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)	19.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)	20.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)	21.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)	22.	(a)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(e)	23.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)	24.	(a)	(b)	(c)	(d)	(e)
11.	(a)	(b)	(c)	(d)	(e)	25.	(a)	(b)	(c)	(d)	(e)
12.	(a)	(b)	(c)	(d)	(e)						
13.	(a)	(b)	(c)	(d)	(e)						
14.	(a)	(b)	(c)	(d)	(e)						

Name: ______________________

Instructor: ______________________

Multiple Choice

1.(6 pts.) The function $f(x) = \frac{1}{2}\ln(1+x) + e^x$ is one-to-one (you do not need to check this). Find the equation of the tangent line to the inverse function $f^{-1}(x)$ at the point $(1, f^{-1}(1))$.

$(1, 0)$ $\quad f'(x) = \frac{1}{2+2x} + e^x$

(a) $y = \frac{2}{3}x - \frac{2}{3}$ (b) $y = \frac{2}{3}x - 1$ (c) $y = 2x - 3$

(d) $y = \frac{3}{2}x + 1$ (e) $y = \frac{1}{2}x + \frac{1}{2}$

$f^{-1}(1) = 0 \quad (f(0) = \frac{1}{2}\ln(1+0) + e^0 = \frac{1}{2}\cancel{\ln 0}^{0} + 1 = 1)$

$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{\frac{1}{2}+1} = \frac{2}{3}$

$y - 0 = \frac{2}{3}(x-1) = \frac{2}{3}x - \frac{2}{3}$

2.(6 pts.) Solve for x in the following equation:

$$\ln(e^x - 1) + \ln(e^x + 1) = 2.$$

(a) $-\frac{3}{3}$ (b) 1 (c) $\frac{e^2+1}{2}$

(d) $\frac{\ln(e^2+1)}{2}$ (e) $\frac{3}{2}$

$\ln(e^x - 1) + \ln(e^x + 1) = \ln[(e^x-1)(e^x+1)] = \ln(e^{2x} - 1) = 2$

$e^{2x} - 1 = e^2 \Rightarrow e^{2x} = e^2 + 1$

$x = \frac{1}{2}\ln(e^2+1)$

Name: ______________________

Instructor: ______________________

$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$

3.(6 pts.) Evaluate the derivative of

$$f(x) = \frac{\ln(\arcsin(x^2))}{2}.$$

(Recall: $\arcsin y = \sin^{-1} y$.)

(a) $\dfrac{1}{2\sqrt{1-x^4}\,(\arcsin(x^2))}$

(b) $\dfrac{1}{(1+[\arcsin(x^2)]^2)}$

(c) $\dfrac{x}{\sqrt{1-x^4}\,(\arcsin(x^2))}$

(d) $\dfrac{x}{(1+x^4)\,(\arcsin(x^2))}$

(e) $\dfrac{-x}{\sin^2(x^2)(\arcsin(x^2))}$

$$f'(x) = \frac{1}{2}\left(\frac{\frac{1}{\sqrt{1-x^4}} \cdot \cancel{2}x}{\arcsin(x^2)}\right) = \frac{x}{\sqrt{1-x^4}\arcsin(x^2)}$$

chain rule

4.(6 pts.) Evaluate the derivative of the function

$$f(x) = (\sin x)^{1/x^2}.$$

(a) $\dfrac{(\sin x)^{(1/x^2)-1}\cos x}{x^2}$

(b) $(\sin x)^{1/x^2}\left[x^2\cos x + 2x\ln(\sin x)\right]$

(c) $(\sin x)^{1/x^2}\left[\dfrac{x^2\cos x - 2x\ln(\sin x)}{x^4}\right]$

(d) $(\sin x)^{1/x^2}\left[\dfrac{x^2\cot x - 2x\ln(\sin x)}{x^4}\right]$

(e) $\dfrac{x^2\cot x - 2x\ln(\sin x)}{x^4}$

$$\ln f = \ln\left[(\sin x)^{1/x^2}\right] = \frac{1}{x^2}\ln(\sin x)$$

$\downarrow d/dx$

$$\frac{f'}{f} = \frac{-2}{x^3}\ln(\sin x) + \frac{\cos x}{x^2 \sin x}$$

$$f' = (\sin x)^{1/x^2}\left[\frac{\cot x}{x^2} - \frac{2\ln(\sin x)}{x^3}\right]$$

Name: ______________________________

Instructor: ___________________________

5. (6 pts.) Evaluate $\int_0^{\frac{\pi}{2}} x \cos x dx$.

$u = x \longrightarrow du = dx$

$dv = \cos x \, dx \rightarrow v = \sin x$

(a) $\frac{\pi}{2} + 1$ (b) $\frac{\pi}{4}$ (c) 0 (d) $\frac{\pi}{2} - 1$ (e) -1

$$\int_0^{\pi/2} x \cos x \, dx = x \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx = (x \sin x + \cos x)\Big|_0^{\pi/2}$$

$$= \left(\frac{\pi}{2} + 0\right) - (0 + 1) = \frac{\pi}{2} - 1$$

6. (6 pts.) Which of the definite integrals shown below is equal to the definite integral

$$\int_0^2 \frac{x^2}{\sqrt{x^2+4}} \, dx?$$

(**Note:** A trigonometric substitution might help.)

(a) $\int_0^{\frac{\pi}{4}} 4\tan^2\theta \sec\theta \, d\theta$

(b) $\int_0^{\frac{\pi}{4}} \frac{2\tan^2\theta}{\sec\theta} \, d\theta$

(c) $\int_0^{\frac{\pi}{2}} \frac{2\tan^2\theta}{\sec\theta} \, d\theta$

(d) $\int_0^{\frac{\pi}{2}} 4\tan^2\theta \sec\theta \, d\theta$

(e) $\int_0^{\frac{\pi}{2}} 4\tan^2\theta \, d\theta$

$x^2 + 4 \sim 4\tan^2\theta + 4 = 4\sec^2\theta \rightsquigarrow \underline{x = 2\tan\theta}$

$0 = 2\tan\theta \rightsquigarrow \theta = 0, \quad 2 = 2\tan\theta \rightsquigarrow \theta = \frac{\pi}{4}$

$$\Rightarrow \int_0^2 \frac{x^2}{\sqrt{x^2+4}} dx = \int_0^{\pi/4} \frac{4\tan^2\theta}{\sqrt{4\tan^2\theta + 4}} (2\sec^2\theta) d\theta = \int_0^{\pi/4} \frac{\overset{4}{\cancel{8}}\tan^2\theta \sec^{\cancel{2}}\theta}{\cancel{2}\cancel{\sec\theta}} d\theta$$

$u = t$

$$= \int_0^{\pi/4} 4\tan^2\theta \sec\theta \, d\theta$$

Name: ______________________

Instructor: ______________________

7.(6 pts.) Evaluate the integral

$$\int_1^e \frac{x^2+x+1}{x(x^2+1)}\,dx.$$

$$\frac{x^2+x+1}{x(x^2+1)} = \frac{1}{x} + \frac{1}{x^2+1}$$

(Recall: $\arctan x = \tan^{-1} x$ and $\arcsin x = \sin^{-1} x$.)

(a) $1 + \arctan(e)$

(b) $1 - \frac{\pi}{2} + \arcsin(e)$

(c) $1 - \frac{\pi}{4} + \arctan(e)$

(d) $1 - \arctan(e)$

(e) $1 - \arcsin(e)$

$$\int_1^e \frac{x^2+x+1}{x(x^2+1)}\,dx = \int_1^e \left(\frac{1}{x} + \frac{1}{x^2+1}\right)dx = (\ln x + \arctan x)\Big|_1^e$$

$$= (\ln e + \arctan e) - (\ln 1 + \arctan 1) = 1 + \arctan e - \left(0 + \frac{\pi}{4}\right)$$

8.(6 pts.) Evaluate the integral

$$\int_0^{\frac{\pi}{2}} \sin^{100} x \cos^3 x\,dx.$$

(a) $\frac{1}{101} - \frac{1}{103}$

(b) $\frac{\pi^{100}}{2^{100}(100)} - \frac{\pi^{102}}{2^{102}(102)}$

(c) $\frac{1}{100} - \frac{1}{102}$

(d) $\frac{\pi^{101}}{2^{101}(101)} - \frac{\pi^{103}}{2^{103}(103)}$

(e) $\frac{1}{101}$

$$\int_0^{\pi/2} \sin^{100} x \cos^3 x\,dx = \int_0^{\pi/2} \sin^{100} x\,(1-\sin^2 x)\cos x\,dx$$

$$\underset{du=\cos x\,dx}{\overset{u=\sin x}{=}} \int_0^1 u^{100}(1-u^2)\,du = \left(\frac{u^{101}}{101} - \frac{u^{103}}{103}\right)\Big|_0^1 = \left(\frac{1}{101} - \frac{1}{103}\right) - (0)$$

Name: ______________________

Instructor: ______________________

9.(6 pts.) Use Simpson's rule with $n = 6$ to approximate the integral

$$\int_0^3 f(x)dx,$$

$\Delta x = \frac{3-0}{6} = 0.5$

where a table of values of $f(x)$ is given below.

x	0	0.5	1	1.5	2	2.5	3
$f(x)$	1	0.5	0.25	0.25	1	1.5	−0.5

Note: The formula sheet may help.

(a) $\frac{13}{3}$ (b) 12 (c) 2 (d) $\frac{13}{6}$ (e) 4

$$\int_0^3 f(x)dx = \frac{0.5}{3}\left(f(0)+4f(0.5)+2f(1)+4f(1.5)+2f(2)+4f(2.5)+f(3)\right)$$

$$= \frac{1}{6}\left(1+2+\cancel{\tfrac{1}{2}}+1+2+6-\cancel{\tfrac{1}{2}}\right) = 2$$

10.(6 pts.)Determine whether the following integral converges or diverges. If it converges, evaluate it.

$$\int_0^\infty \frac{e^x}{1+e^{2x}}dx.$$

(a) $\frac{\pi}{2}$ (b) The integral diverges.

(c) 1 (d) $\frac{\pi}{4}$

(e) 2

$$\int_0^\infty \frac{e^x}{1+e^{2x}}dx \overset{u=e^x}{=\!=} \int_1^\infty \frac{1}{1+u^2}du = \lim_{t\to\infty}\int_1^t \frac{1}{1+u^2}du = \lim_{t\to\infty} \arctan u\Big|_1^t$$

$$= \lim_{t\to\infty}\left(\arctan t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4}$$

Name: ______________________

Instructor: ______________________

11.(6 pts.)The length of the curve $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$, $\frac{1}{2} \le x \le 1$, is given by:

(a) $\dfrac{1}{2}\displaystyle\int_{1/2}^{1} \sqrt{1 + (x^2 + x^{-2})^2}\,dx$

(b) $\dfrac{1}{2}\displaystyle\int_{1/2}^{1} \sqrt{1 + (x + x^{-1})^2}\,dx$

(c) $\dfrac{1}{2}\displaystyle\int_{1/2}^{1} (x + x^{-1})\,dx$

(d) $\dfrac{1}{2}\displaystyle\int_{1/2}^{1} (x^2 + x^{-2})\,dx$

(e) $\dfrac{1}{2}\displaystyle\int_{1/2}^{1} \sqrt{(x^2 + x^{-2})}\,dx$

$$y' = \frac{x^2}{2} - \frac{1}{2x^2} \Rightarrow (y')^2 = \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}$$

$$L = \int_{1/2}^{1} \sqrt{1+(y')^2}\,dx = \int_{1/2}^{1} \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}}\,dx = \int_{1/2}^{1} \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\,dx$$

$$= \int_{1/2}^{1} \frac{1}{2}(x^2 + x^{-2})\,dx$$

12.(6 pts.)The solution to the initial value problem

$$x\frac{dy}{dx} + 2y = e^{x^2} \qquad y(1) = 0$$

is

(a) $y = \dfrac{e^{x^2} - e}{2x^2}$

(b) $y = \dfrac{e^{x} - e}{2x^2}$

(c) $y = \dfrac{e^{x} - e}{2x}$

(d) $y = \dfrac{e^{x^2} - e}{2x}$

(e) $y = xe^x - e$

$$\Rightarrow y' + \frac{2}{x}y = \frac{1}{x}e^{x^2} \qquad \mu = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$y = \frac{1}{x^2}\int x^2\left(\frac{1}{x}e^{x^2}\right)dx = \frac{1}{x^2}\underbrace{\int xe^{x^2}dx}_{u=x^2,\ du=2xdx} = \frac{1}{x^2}\left(\frac{1}{2}e^{x^2} + C\right)$$

$$y(1) = \left(\frac{1}{2}e + C\right) = 0 \qquad C = -\frac{e}{2} \qquad \Rightarrow y = \frac{e^{x^2} - e}{2x^2}$$

Name: ______________________

Instructor: ______________________

13.(6 pts.)The solution to the initial value problem

$$y' = x\cos^2 y \qquad\qquad y(2) = 0$$

satisfies the implicit equation

(a) $\cos y = \frac{x^2}{2} - 2$

(b) $\ln|\sec(y) + \tan(y)| = \frac{x^2}{2} - 2$

(c) $\tan y = \frac{x^2}{2} - 2$

(d) $\sin(2y) = 4x - 8$

(c) $\arctan y = \frac{x^2}{2} - 2$

$y' = x\cos^2 y \rightsquigarrow \sec^2 y\, y' = x \rightsquigarrow \tan y = \frac{1}{2}x^2 + C$

$y(2) = 0 \rightsquigarrow \underbrace{\tan 0}_{=0} = \frac{1}{2}(2)^2 + C = 2 + C \Rightarrow C = -2$

$\tan y = \frac{1}{2}x^2 - 2$

14.(6 pts.)Consider the initial value problem

$$\begin{cases} y' &= \sin[\pi(x+y)] \\ y(0) &= 0. \end{cases}$$

Use Euler's method with two steps of step size $\underset{=h}{0.5}$ to find an approximate value of $y(1)$.
Note: The formula sheet may help.

(a) 0 (b) 0.5 (c) -1 (d) 1 (e) -0.5

$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) \quad (y' = f(x,y))$

n	0	1	2
x_n	0	1/2	1
y_n	0	0	1/2

$y_1 = 0 + \frac{1}{2}\sin(0) = 0$

$y_2 = 0 + \frac{1}{2}\sin\left(\pi\left(\frac{1}{2}+0\right)\right) = \frac{1}{2}$

Name: ______________________

Instructor: ______________________

15.(6 pts.) Consider the following **sequences**:

$$(I)\ \left\{(-1)^n \frac{n-1}{2n^2+1}\right\}_{n=1}^{\infty} \qquad (II)\ \left\{\frac{n^2-1}{\ln(n)}\right\}_{n=1}^{\infty} \qquad (III)\ \left\{3^{1/n}\right\}_{n=1}^{\infty}$$

Which of the following statements is true?

~~(a) All three sequences diverge.~~

~~(b) Sequence III diverges and sequences I and II converge.~~

(c) Sequence II diverges and sequences I and III converge.

~~(d) All three sequences converge.~~

~~(e) Sequence I diverges and sequences II and III converge.~~

(II) $n^2-1 \geq \ln(n)$ for $n \geq 1 \Rightarrow$ diverges : eliminates b, d, e

(III) $\lim_{n\to\infty} 3^{1/n} = 3^{\lim_{n\to\infty} 1/n} = 3^0 = 1 \Rightarrow$ conv. : eliminates a

16.(6 pts.) Find $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{3^{n-1}}$.

(a) $\frac{3}{2}$ (b) $-\frac{9}{2}$ (c) 2 (d) 3 (e) $\frac{15}{2}$

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^{n-1}} = \sum_{n=1}^{\infty}\left(\frac{1^{n-1}}{3^{n-1}} + \frac{2\cdot 2^{n-1}}{3^{n-1}}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1} + \sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1}$$

$$= \frac{1}{1-\frac{1}{3}} + \frac{2}{1-\frac{2}{3}} = \frac{3}{2} + 6 = \frac{15}{2}$$

Name: ______________________

Instructor: ______________________

17.(6 pts.) Consider the following series

$$(I)\ \sum_{n=3}^{\infty}\frac{(-1)^n}{\ln(n)} \qquad (II)\ \sum_{n=2}^{\infty}\frac{2n^2}{n^4+1} \qquad (III)\ \sum_{n=1}^{\infty}\frac{n!}{3^n}$$

Which of the following statements is true?

(a) Only III converges

(b) Only I and II converge

(c) Only II and III converge

(d) All three diverge

(e) All three converge

(I) alt. ser.

ⓐ $\lim_{n\to\infty}\frac{1}{\ln(n)}=0$ $\Rightarrow$ conv.

ⓑ $\ln(n)$ inc. $\Rightarrow \frac{1}{\ln(n)}$ dec

(II) lim comp w/ $\sum_{n=2}^{\infty}\frac{1}{n^2}$ (conv.)

$\lim_{n\to\infty}\left|\frac{2n^2}{n^4+1}\cdot\frac{n^2}{1}\right|=2<\infty$

$\Rightarrow$ conv

(III) $n!>3^n$ eventually

$\Rightarrow \lim_{n\to\infty}\frac{n!}{3^n}=\infty$

$\Rightarrow$ div

18.(6 pts.) Consider the following series

$$(I)\ \sum_{n=3}^{\infty}\frac{\sin(\frac{1}{n})}{\frac{1}{n}} \qquad (II)\ \sum_{n=3}^{\infty}\frac{(-1)^n}{\sqrt[3]{n}-1}.$$

Which of the following statements is true?

(a) (I) is divergent and (II) is conditionally convergent.

~~(b)~~ (I) is absolutely convergent and (II) diverges.

~~(c)~~ (I) and (II) are both conditionally convergent.

~~(d)~~ (I) is absolutely convergent and (II) is conditionally convergent.

(e) (I) and (II) both diverge.

(I) $\lim_{n\to\infty}\frac{\sin(\frac{1}{n})}{\frac{1}{n}}=\lim_{m\to 0}\frac{\sin(m)}{m}=1\neq 0$ div.

eliminates: b, c, d

(II) alt series:

ⓐ $\lim_{n\to\infty}\frac{1}{\sqrt[3]{n-1}}=0$ ✓ ⓑ $\sqrt[3]{n-1}$ increasing $\Rightarrow \frac{1}{\sqrt[3]{n-1}}$ dec. $\Rightarrow$ conv.

10

Name: ____________________

Instructor: ____________________

19.(6 pts.) Find the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{5^n(n+1)}.$$

(a) $(-5,5]$ (b) $(-\frac{1}{5},\frac{1}{5}]$ (c) $[-5,5)$ (d) $(-1,1]$ (e) $[-\frac{1}{5},\frac{1}{5})$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{5^{n+1}(n+2)}\cdot\frac{5^n(n+1)}{x^n}\right| = \lim_{n\to\infty}\frac{n+1}{5(n+2)}|x| = \frac{|x|}{5} < 1 \Rightarrow |x|<5$$

$x=-5$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}$ ✓ conv. by alt. series

$\boxed{x=5}$ $\sum_{n=1}^{\infty}\frac{1}{n+1}$ X div. by lim. comp. to $\Sigma\frac{1}{n}$

$[-5,5)$

20.(6 pts.) Find a power series representation for the the function $f(x)=\ln(1+x^3)$ on the interval $-1<x<1$.

Hint: $\frac{d}{dx}(\ln(1+x^3)) = \frac{3x^2}{1+x^3}$.

(a) $\sum_{n=0}^{\infty}\frac{x^{n+3}}{n+4}$ (b) $\sum_{n=0}^{\infty}(-1)^n x^{3n}$ (c) $\sum_{n=0}^{\infty}x^{3n}$

(d) $\sum_{n=0}^{\infty}\frac{(-1)^n 3x^{3n+1}}{3n+1}$ (e) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+3}}{n+1}$

$$\frac{3x^2}{1+x^3} = \frac{3x^2}{1-(-x^3)} = \sum_{n=0}^{\infty}3x^2(-x^3)^n = \sum_{n=0}^{\infty}(-1)^n 3x^{3n+2}$$

$$\ln(1+x^3) = \int\frac{3x^2}{1+x^3}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n \cancel{3}x^{3n+3}}{\cancel{3}n+\cancel{3}} + C = \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+3}}{n+1} + C$$

$$x=0:\ 0=\ln(1+0^3) = \sum_{n=0}^{\infty}\frac{(-1)^n 0}{n+1} + C = C$$

Name: ______________________

Instructor: ______________________

21.(6 pts.) Which of the power series given below is the McLaurin series (i.e. Taylor series at $a = 0$) of the function

$$f(x) = xe^{x^3}?$$

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$. (b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$. (c) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n)!}$.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}$. (e) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(n)!}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$xe^{x^3} = x\sum_{n=0}^{\infty}\frac{(x^3)^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{3n+1}}{n!}$$

22.(6 pts.) Which of the polynomials shown below is the third Taylor polynomial of $f(x) = \dfrac{1}{(1-x)^2}$ at $a = -1$?

want $(x+1)^n$

Notice: $\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right) = \dfrac{1}{(1-x)^2}$

(a) $1 + 2x + 3x^2 + 4x^3$

(b) $\dfrac{1}{2^2} + \dfrac{2}{2^3}x + \dfrac{3}{2^4}x^2 + \dfrac{4}{2^5}x^3$

(c) $\dfrac{1}{2^2} + \dfrac{2!}{2^3}(x+1) + \dfrac{3!}{2^4}(x+1)^2 + \dfrac{4!}{2^5}(x+1)^3$

(d) $\dfrac{1}{2^2} + \dfrac{2}{2^3}(x+1) + \dfrac{3}{2^4}(x+1)^2 + \dfrac{4}{2^5}(x+1)^3$

(e) $1 + (x+1) + (x+1)^2 + (x+1)^3$

$$\frac{1}{1-x} = \frac{1}{1-(x+1-1)} = \frac{1}{2-(x+1)} = \frac{1/2}{1-\left(\frac{x+1}{2}\right)} = \sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{x+1}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(x+1)^n}{2^{n+1}}$$

$$= \cancel{\tfrac{1}{2}} + \frac{(x+1)}{2^2} + \frac{(x+1)^2}{2^3} + \frac{(x+1)^3}{2^4} + \frac{(x+1)^4}{2^5} + \cdots$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \underbrace{\frac{1}{2^2} + \frac{2(x+1)}{2^3} + \frac{3(x+1)^2}{2^4} + \frac{4(x+1)^3}{2^5}}_{=T_3(x)} + \cdots$$

Name: ____________________

Instructor: ____________________

23.(6 pts.) Find the length of the parameterized curve

$$x = \cos^2 t, \qquad y = \sin^2 t, \quad 0 \le t \le \frac{\pi}{2}.$$

$x' = -2\sin t\cos t$
$y' = 2\sin t\cos t$

(a) $\frac{1}{2}$ (b) $\sqrt{2}$ (c) 1 (d) $\frac{\pi}{2}$ (e) $\frac{1}{2\sqrt{2}}$

$$L = \int_0^{\pi/2} \sqrt{(x')^2 + (y')^2}\,dt = \int_0^{\pi/2} \sqrt{4\cos^2 t\sin^2 t + 4\cos^2 t\sin^2 t}\,dt$$

$$= \int_0^{\pi/2} 2\sqrt{2}\,\sin t\cos t\,dt \overset{u=\sin t}{=\!=} 2\sqrt{2}\int_0^1 u\,du = \sqrt{2}\,u^2\Big|_0^1 = \sqrt{2}$$

24.(6 pts.) The point $(2, \frac{7\pi}{3})$ in polar coordinates corresponds to which point below in Cartesian coordinates?

(a) $(1, \sqrt{3})$ (b) $(-1, \sqrt{3})$

(c) $(\sqrt{3}, 1)$ (d) $(-\sqrt{3}, 1)$

(e) Since $\frac{7\pi}{3} > 2\pi$, there is no such point

$$\frac{7\pi}{3} \sim \frac{\pi}{3}$$

$$x = 2\cos\frac{7\pi}{3} = 2\left(\frac{1}{2}\right) = 1$$

$$y = 2\sin\frac{7\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

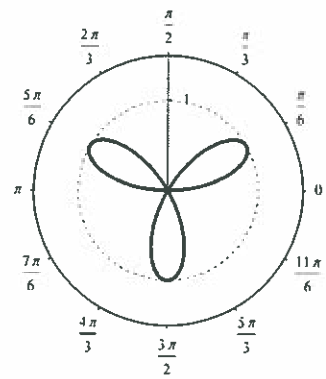
Name: ______________________

Instructor: ______________________

25.(6 pts.) Find the area of the region enclosed by the polar curve

$$r = \sin(3\theta), \quad 0 \le \theta \le \pi.$$

(**Note:** The formula sheet may help here.)



(a) 3 (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$ (e) π

$0 \le \theta \le \frac{\pi}{3}$ encloses one petal:

$$\text{Area} = 3 \cdot \int_0^{\pi/3} \frac{r^2}{2}\, d\theta = \frac{3}{2}\int_0^{\pi/3} \sin^2(3\theta)\, d\theta$$

$$= \frac{3}{4}\int_0^{\pi/3} (1 - \cos 6\theta)\, d\theta = \frac{3}{4}\left(\theta - \frac{1}{6}\sin 6\theta\right)\Big|_0^{\pi/3}$$

$$= \frac{3}{4}\left[\left(\frac{\pi}{3} - \cancel{\frac{1}{6}\sin 2\pi}\right) - \cancel{\left(0 - \frac{1}{6}\sin 0\right)}\right] = \frac{\pi}{4}$$

Name: ______________________

Instructor: ______________________

The following is the list of useful trigonometric formulas:

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x-y) + \sin(x+y)\big)$$

$$\sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\int \csc\theta d\theta = \ln|\csc\theta - \cot\theta| + C$$

$$\int \csc^2\theta d\theta = -\cot x + C$$

Trapezoidal Rule If f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + +2f(x_{n-1}) + f(x_n))$$

Error Bounds If $|f''(x)| \le K$ for $a \le x \le b$. Let E_T denote the error for the trapezoidal approximation then $|E_T| \le \frac{K(b-a)^3}{12n^2}$

Simpson's rule If f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n))$$

Error Bound for Simpson's Rule Suppose that $|f^{(4)}(x)| \le K$ for $a \le x \le b$. If E_S is the error involved in using Simpson's Rule, then $|E_S| \le \frac{K(b-a)^5}{180n^4}$

Euler's Method with step size h: $y_i = y_{i-1} + hF(x_{i-1}, y_{i-1})$, $\quad x_i = x_0 + ih$.

Name: ____________________

Math 10560, Final Exam:
May 7, 2014

Instructor: ANSWERS

- The Honor Code is in effect for this examination, including keeping your answer sheet under cover.
- No **calculators** should be used during the exam.
- **Turn off and put away all cell-phones** and similar electronic devices.
- **Head phones** are not allowed.
- **Put away all notes and formula sheets** where they cannot be viewed.
- The exam lasts for two hours.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 15 pages of the test.
- Hand in the entire exam.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(•)	(b)	(c)	(d)	(e)	15.	(a)	(b)	(•)	(d)	(e)
2.	(a)	(b)	(c)	(•)	(e)	16.	(a)	(b)	(c)	(d)	(•)
3.	(a)	(b)	(•)	(d)	(e)	17.	(a)	(•)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(•)	(e)	18.	(•)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(•)	(e)	19.	(a)	(b)	(•)	(d)	(e)
6.	(•)	(b)	(c)	(d)	(e)	20.	(a)	(b)	(c)	(d)	(•)
7.	(a)	(b)	(•)	(d)	(e)	21.	(a)	(b)	(c)	(d)	(•)
8.	(•)	(b)	(c)	(d)	(e)	22.	(a)	(b)	(c)	(•)	(e)
9.	(a)	(b)	(•)	(d)	(e)	23.	(a)	(•)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(•)	(e)	24.	(•)	(b)	(c)	(d)	(e)
11.	(a)	(b)	(c)	(•)	(e)	25.	(a)	(b)	(•)	(d)	(e)
12.	(•)	(b)	(c)	(d)	(e)						
13.	(a)	(b)	(•)	(d)	(e)						
14.	(a)	(•)	(c)	(d)	(e)						